Preface

Mathematical Reasoning: Writing and Proof is designed to be a text for the first course in the college mathematics curriculum that introduces students to the processes of constructing and writing proofs and focuses on the formal development of mathematics. The primary goals of the text are as follows:

- To help students learn how to read and understand mathematical definitions and proofs;
- To help students learn how to construct mathematical proofs;
- To help students learn how to write mathematical proofs according to accepted guidelines so that their work and reasoning can be understood by others; and
- To provide students with mathematical material that will be needed for their further study of mathematics.

This type of course is becoming a standard part of the mathematics major at most colleges and universities. It is often referred to as a "transition course" from the calculus sequence to the upper-level courses in the major. The transition is from the problem-solving orientation of calculus to the more abstract and theoretical upper-level courses. This is needed today because many students complete their study of calculus without seeing a formal proof or having constructed a proof of their own. This is in contrast to many upper-level mathematics courses, where the emphasis is on the formal development of abstract mathematical ideas, and the expectations are that students will be able to read and understand proofs and be able to construct and write coherent, understandable mathematical proofs.

Important Features of the Book

Mathematical Reasoning: Writing and Proof was written to assist students with the transition from calculus to upper-level mathematics courses. Students should be able to use this text with a background of one semester of calculus. Following are some of the important ways this text will help with this transition.

1. **Emphasis on Writing in Mathematics**

   Issues dealing with writing mathematical exposition are addressed throughout the book. Guidelines for writing mathematical proofs are incorporated into the book. These guidelines are introduced as needed and begin in Chapter 1. Appendix A contains a summary of all the guidelines for writing mathematical proofs that are introduced throughout the text. In addition, every attempt has been made to ensure that every completed proof presented in this text is written according to these guidelines. This provides students with examples of well-written proofs.

2. **Instruction in the Process of Constructing Proofs**

   One of the primary goals of this book is to develop students' abilities to construct mathematical proofs. Another goal is to develop their abilities to write the proof in a coherent manner that conveys an understanding of the proof to the reader. These are two distinct skills.

   Instruction on how to write proofs begins in Section 1.2 and is developed further in Chapter 3. In addition, Chapter 5 is devoted to developing students' abilities to construct proofs using mathematical induction.

   Students are introduced to a method to organize their thought processes when attempting to construct a proof that uses a so-called know-show table. (See Section 1.2 and Section 3.1.) Students use this table to work backward from what it is they are trying to prove while at the same time working forward from the assumptions of the problem. The know-show tables are used quite extensively in Chapters 1 and 3. However, the explicit use of know-show tables is gradually reduced and these tables are rarely used in the later chapters. One reason for this is that these tables may work well when there appears to be only one way of proving a certain result. As the proofs become more complicated or other methods of proof (such as proofs using cases) are used, these know-show tables become less useful.

   So the know-show tables are not to be considered an absolute necessity in using the text. However, they are useful for students beginning to learn how to construct and write proofs. They provide a convenient way for students to organize their work. More importantly, they introduce students to a way of thinking about a problem. Instead of immediately trying to write a complete proof, the know-show table forces students to stop, think, and ask questions such as

- Just exactly what is it that I am trying to prove?
- How can I prove this?
- What methods do I have that may allow me to prove this?
- What are the assumptions?
- How can I use these assumptions to prove the result?

Being able to ask these questions is a big step in constructing a proof. The next task is to answer the questions and to use those answers to construct a proof.

3. Emphasis on Active Learning

One of the underlying premises of this text is that the best way to learn and understand mathematics is to be actively involved in the learning process. However, it is unlikely that students will learn all the mathematics in a given course on their own. Students actively involved in learning mathematics need appropriate materials that will provide guidance and support in their learning of mathematics. This text provides these opportunities by

- Incorporating two or three Preview Activities at the beginning of each section. These Preview Activities should be completed by the students prior to the classroom discussion of the section. The purpose of the Preview Activities is to prepare students to participate in the classroom discussion of the section. Some Preview Activities will review prior mathematical work that is necessary for the new section. This prior work may contain material from previous mathematical courses or it may contain material covered earlier in this text. Other preview activities will introduce new concepts and definitions that will be used when that section is discussed in class.
- Including several Progress Checks throughout each section. These are either short exercises or short activities designed to help the students determine if they are understanding the material as it is presented. Some progress checks are also intended to prepare the student for the next topic in the section. Answers to the Progress Checks are provided at the end of each chapter.
- Including activities at the end of each section. These activities can be done individually or in a collaborative learning setting, where students work in groups to brainstorm, make conjectures, test each others' ideas, reach consensus, and, it is hoped, develop sound mathematical arguments to support their work. These activities can also be assigned as homework in addition to the exercises at the end of each section.

Content and Organization

Mathematical content is needed as a vehicle for learning how to construct and write proofs. The mathematical content for this text is drawn primarily from elementary number theory, including congruence arithmetic; elementary set theory; functions, including injections, surjections, and the inverse of a function; relations and equivalence relations; further topics in number theory such as greatest common divisors and prime factorizations; and cardinality of sets, including countable and uncountable sets. This material was chosen because it can be used to illustrate a broad range of proof techniques and it is needed as a prerequisite for many upper-level mathematics courses.

The chapters in the text can roughly be divided into the following classes:

- Constructing and Writing Proofs: Chapters 1, 3, and 5
- Mathematical Content: Chapters 4, 6, 7, 8, and 9
- Logic: Chapter 2

The first chapter sets the stage for the rest of the book. It introduces students to the use of conditional statements in mathematics, begins instruction in the process of constructing a direct proof of a conditional statement, and introduces many of the writing guidelines that will be used throughout the rest of the book. This is not meant to be a thorough introduction to methods of proof. Before this is done, it is necessary to introduce the students to the parts of logic that are needed to aid in the construction of proofs. This is done in Chapter 2.

Students need to learn some logic and gain experience in the traditional language and proof methods used in mathematics. Since this is a text that deals with constructing and writing mathematical proofs, the logic that is presented in Chapter 2 is intended to aid in the construction of proofs. The goals are to provide students with a thorough understanding of conditional statements, quantifiers, and logical equivalencies. Emphasis is placed on writing correct and useful negations of statements, especially those involving quantifiers. The logical equivalencies that are presented provide the logical basis for some of the standard proof techniques, such as proof by contrapositive, proof by contradiction, and proof using cases.

The standard methods for mathematical proofs are discussed in detail in Chapter 3. The mathematical content that is introduced to illustrate these proof methods includes some elementary number theory, including congruence arithmetic. These concepts are used consistently throughout the text as a way to demonstrate ideas in direct proof, proof by contrapositive, proof by contradiction, proof using cases, and proofs using mathematical induction. This gives students a strong introduction to important mathematical ideas while providing the instructor a consistent reference point and an example of how mathematical notation can greatly simplify a concept.

Chapter 4 provides a break from introducing new proof techniques. Concepts of set theory are introduced, and the methods of proof studied in Chapter 3 are used to prove results about sets and operations on sets. The idea of an "element-chasing proof" is introduced in Section 4.2.

The three sections of Chapter 5 are devoted to proofs using mathematical induction. Again, the emphasis is not only on understanding mathematical induction but also on developing the ability to construct and write proofs that use mathematical induction.

The last four chapters are considered "mathematical content" chapters. Chapter 6 provides a thorough study of functions. Functions are studied before relations in order to begin with the more specific notion with which students have some familiarity and move toward the more general notion of a relation. The concept of a function is reviewed but with attention paid to being precise with terminology and is then extended to the general definition of a function. Various proof techniques are employed in the study of injections, surjections, composition of functions, inverses of functions, and functions acting on sets.

Chapter 7 introduces the concepts of relations and equivalence relations. Section 7.4 is included to provide a link between the concept of an equivalence relation and the number theory that has been discussed throughout the text.

Chapter 8 continues the study of number theory. The highlights include problems dealing with greatest common divisors, prime numbers, the Fundamental Theorem of Arithmetic, and linear Diophantine equations.

Finally, Chapter 9 deals with further topics in set theory, focusing on cardinality, finite sets, countable sets, and uncountable sets.

Designing a Course

Most instructors who use this text will design a course specifically suited to their needs and the needs of their institution. However, a standard one-semester course in constructing and writing proofs could cover the first six chapters of the text and at least one of Chapter 7, Chapter 8, or Chapter 9. A class consisting of well-prepared and motivated students could cover two of the last three chapters. If either of these options is a bit too ambitious, Sections 4.5, 5.3, 6.6, 7.4, and 8.3 can be considered optional sections. These are interesting sections that contain important material, but the content of these sections is not used in the rest of the book.

Changes in the Second Edition

Changes to the Pedagogy and Format of the Book

1. There are now several Progress Checks throughout each section. These are either short exercises or short activities designed to help the student determine if he or she is understanding the material as it is presented. Answers to the Progress Checks are provided at the end of each chapter. Many of the shorter activities from the first edition are now Progress Checks, and several new Progress Checks have been written.

2. Several additional exercises have been written for the book. This was done at the suggestion of several users of the first edition who desired a greater variety of exercises and types of exercises.

3. Several sections of the text now include exercises called Evaluation of Proofs. (The first such exercise appears in Section 3.1.) For these exercises, there is a proposed proof of a proposition. However, the proposition may be true or may be false. If a proposition is false, the proposed proof is, of course, incorrect, and the student is asked to find the error in the proof and then provide a counterexample showing that the proposition is false. However, if the proposition is true, the proof may be incorrect or not well written. In keeping with the emphasis on writing, students are then asked to correct the proof and/or provide a well-written proof according to the guidelines established in the book.

4. To assist students with studying the material in the text, there is now a summary at the end of each chapter. The summaries usually list the important definitions introduced in the chapter and the important results proven in the chapter. If appropriate, the summary also describes the important proof techniques discussed in the chapter.

5. More answers to exercises are included in an appendix. This was done in response to suggestions from many students at Grand Valley and some students from other institutions who were using the book. In addition, those exercises with an answer or a hint in the appendix are now preceded by a star ($\star$).

Changes in Content and Organization

Chapter 1. The list of writing guidelines in Chapter 1 has been shortened. Most of the writing guidelines are in Sections 1.2 and 3.1. More information about the closure properties of the number systems has been added to Section 1.1. These are good examples of the use of conditional statements, which is an important part of Section 1.1. In addition, the definition of a conditional statement has been changed so that sentences such as "If n is a positive integer, then $n^2 - n + 41$ is a prime number" can be considered to be a statement. The definition now states that a **conditional statement** is a statement that can be written in the form "If P then Q," where P and Q are sentences.

Chapter 2. There is more emphasis on the use of set builder notation in Section 2.3, and the order of the sections has been changed to

Section 2.1 – Statements and Logical Operators (old Section 2.2)
Section 2.2 – Logically Equivalent Statements (old Section 2.3)
Section 2.3 – Predicates, Sets, and Quantifiers (old Section 2.1)
Section 2.4 – Quantifiers and Negations (old Section 2.4)

Chapter 3. The definition of "divides" has been changed so that the "divisor" must be nonzero. In the first edition, it was stated that an integer m divides an integer n provided that there exists an integer q such that $n = m \cdot q$. This leads to the conclusion that 0 divides 0, and this bothered some students (and some faculty). So the new definition states that a nonzero integer m divides an integer n provided that there exists an integer q such that $n = m \cdot q$.

Section 3.5 (from the first edition) on constructive (and nonconstructive) proofs has been eliminated, but the material has been simplified and incorporated in Section 3.2 (More Methods of Proof). The old Section 3.4 (Using Cases in Proofs) has been divided into two sections. The new Section 3.4 is still "Using Cases in Proofs," and it includes new material on absolute value. The new Section 3.5 is "The Division Algorithm and Congruence." This new section contains more about these topics than the first edition.

Chapter 4. A new section on indexed families of sets has been added as the last section in the chapter. This can be considered an optional section.

Chapter 5. The term "inductive set" is now defined in Preview Activity 3 in Section 5.1. In the first edition, this was just called Property I in the preview and then inductive set was defined in the section. Activity 5.2 (The Importance of the Basis Step) seemed to cause confusion among students, and so this activity has been eliminated. The importance of the basis step is now discussed in the subsection called "Some Comments about Mathematical Induction."

Chapter 6. Some material has been added in Section 6.1 that will help students understand why we use both the codomain of a function and the range of a function and what is the distinction between these two sets. The material on arrow diagrams is now a subsection of Section 6.1 instead of an example. In addition, continued work with set notation has been reinforced in this section by having students determine the set of all preimages of an element in the codomain rather than determining all the preimages of the element.

Section 6.2 has been significantly rewritten. The order of the preview activities has been changed to better match the order of the material presented in the section. The subsection "Some Standard Functions" has been eliminated, but much of this material has been moved to the exercises. A progress check dealing with the average value of a finite set of numbers has been added to the subsection dealing with mathematical processes as functions. Finally, the material dealing with a function as a set of ordered pairs has been moved to a preview activity in Section 6.5.

Preview Activity 2 in Section 6.3 has been changed. The old Preview Activity 2 was also an activity in Section 6.1. The new preview activity introduces the statements that are used to define an injection and a surjection and asks students to work with these statements for particular functions. This should be very helpful in the discussion of the material in Section 6.3.

Preview Activities 1 and 2 in Section 6.5 are new and deal with describing a function as a set of ordered pairs, and finally, Section 9.1 (Functions Acting on Sets) from the first edition has been moved to Chapter 6. It is now Section 6.6.

Chapter 7. Several new exercises have been added to Chapter 7. The preview activities in Section 7.1 have been rewritten. The former Preview Activity 3 is now Preview Activity 1 and a new preview activity has been added.

The two relations used in Preview Activity 2 in Section 7.2 did not seem to work well. The second edition now uses two new relations. Students are asked to determine if these relations are reflexive, symmetric, or transitive.

Chapter 8. The proof of the theorem that states that there are infinitely many primes is now an activity at the end of Section 8.2. Some linear Diophantine equations in Section 8.3 that were in examples are now part of a progress check. This will give students an opportunity to try to solve these equations on their own.

Chapter 9. As noted earlier, Section 9.1 from the first edition is now Section 6.6. The preview activities in the new Section 9.1 have been rewritten. The major change is that the proof that the open interval $(0, 1)$ is equivalent to the open interval $(0, b)$ is part of a progress check. The new example in Preview Activity 1 is to show that $\mathbb{R}$ is equivalent to $\mathbb{R}^+$ using the exponential function.

A new preview activity has been added to Section 9.2 (which was Section 9.3 in the first edition). This preview is essentially Activity 9.23 on page 388 of the first edition.

Supplementary Materials for the Instructor

The instructor's manual for this text includes suggestions on how to use the text, how to incorporate writing into the course, and how to use the preview activities and activities. The manual also includes solutions for all of the preview activities, activities, and exercises. In addition, for each section, there is a description of the purpose of each preview activity and how it is used in the corresponding section, and there are suggestions about how to use each activity in that section. The intention is to make it as easy as possible for the instructor to use the text in an active learning environment. These activities can also be used in a more traditional lecture-discussion course. In that case, some of the activities would be discussed in class or assigned as homework.

PDF files are also available to instructors who use the text to assist them in posting solutions to a course Web page or distributing printed solutions to students. For each section, there is a file containing the solutions of the preview activities, and for each activity in the text, there is a file containing the solutions for that activity. Instructors can contact the author through his e-mail address (sundstrt@gvsu.edu) for access to the files.

In addition, all instructor resources can be downloaded from the Prentice Hall Web site, www.prenhall.com. Select "Browse our catalog," then click on "Mathematics"; select your course and choose your text. Under "Resources," on the left side, select "instructor" and choose the supplement you need to download. You will be required to run through a one-time registration before you can complete this process.

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Ted Sundstrom
sundstrt@gvsu.edu